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ABSTRACT

Mathematics is thought to be the most culturally independent of all academic subjects. "New Math" textbooks printed in the United States or Belgium were translated into Spanish and Portuguese with only minor variations in the story problems and are now taught in most Latin-American countries. Looking backwards, it was not different in past years in Colombia, where standard school textbooks copied each other in a chain going back to Spanish and Latin Renaissance arithmetics. The myth of mathematics as the universal language of science and the superficial image of mathematical truth as invariable and "a priori" structured in human reason, reinforced the stereotype of mathematics as a supra-cultural subject. The purpose of this paper is to (1) determine as accurately as possible the cultural dependence of the learning process in mathematics; (2) examine what is essentially right in the claim to universality in mathematics; (3) consider where the culturally specific aspects of mathematics are to be located, both theoretically and empirically; and (4) determine how they are to be used to develop a culturally adapted curriculum for elementary school mathematics. (JN)

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Learning Elementary School Mathematics as a Culturally Conditioned Process

by

Carlos E. Vasco

2/17/84

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"Learning Elementary School Mathematics as a Culturally Conditioned Process."

By: Carlos E. Vasco

1. Introduction

Norma X lives in a slum up the steep hills to the East of Bogota, Colombia. When I met her six years ago, she was seven years old and was attending first grade at the local school. She used to help her mother sell food and soft drinks on Sundays at a nearby park. She could sell a dozen bottles of soft drink at 9 pesos a bottle and quickly and accurately return change from a 200-peso bill. But she flunked first-grade arithmetic.

Luis Eduardo Y dropped out of the 4th grade in 1975. Some years later he wanted to pass the five-grade elementary school through a State examination, which is given twice a

year to prospective high-school entrants. He did not dare take it, because he thought he knew nothing about fractions. But I knew that he was an able mechanic and had very good experience with nuts and bolts, wrenches and calipers. He could "see" in a flash of his alert mind that a 3/8" jaw would not bite a half-inch nut or even a 1-cm European bolt-head. I needed paper and pencil to verify that he was right. Once he realized that he knew a good deal about fractions, he had no trouble taking and passing the State examination with flying colors. Now he has just finished High School and wants to be a teacher (of mathematics, of course).

Mathematics is thought to be the most culturally independent of all academic subjects. "New Math" textbooks printed in the U.S. or Belgium were translated into Spanish and Portuguese with only minor variations in the story-problems and are now taught in most Latin-American countries. Looking backwards, it was not different in the years past: standard school textbooks used in Colombia copied each other in a chain going back to Spanish and Latin Renaissance arithmetics. The myth of mathematics as the universal language of science and the superficial image of mathematical truth as invariably and "a priori" structured in human reason, reinforced the stereotype of mathematics as a supra-cultural subject.

The distinction between Arabic and Roman numerals signaled a degree of cultural dependence. But it was thought to be due to the early stages of development of primitive peoples, to historical accident, and to the late development of writing. It was something archaic, located at the superficial level of symbolism, and wholly immaterial from the viewpoint of modern mathematics.

But the word "archaic" points deeper than its superficial connotation of things obsolete. It points to the origins, to the sources, to the the princes, the principles and the principals. Should we look deeper into the roots of mathematics?

Michel de Foucault's Archeology of Knowledge taught me to discover hidden strata through a close analysis of discourse. Jean Piaget's truly archeologic studies of the slow and tortuous development of logico-mathematical concepts in children taught me to listen to them very carefully and to interpret their apparent errors as symptoms of a different type of logical structure. The Russian Psychologist Lev A. Vygotsky explicitly taught me where to start the search:

> That children's learning begins long before they attend school is the starting point of this discussion. Any learning a child encounters in school always has a previous history. For example, children begin to study arithmetic in school, but long beforehand they have had some experience with quantity--they have had to deal with operations of division, addition, subtraction, and determination

of size. Consequently, children have their own preschool arithmetic, which only myopic psychologists could ignore (1).

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(1) Quoted with permission from: VYGOTSKY, L.S., Mind in Society. The Development of Higher Psychological Processes. Harvard University Press, Cambridge - London, 1978, p. 84.

The purpose of this paper is to determine as accurately as possible the cultural dependence of the learning process in mathematics. What is essentially right in the claim to universality in mathematics? Where are the culturally specific aspects of mathematics to be located, both theoretically and empirically? How are they to be used to develop a culturally adapted curriculum for elementary school mathematics? These are the questions that guide the search.

2. The Cultural Setting.

In Colombia, grade-school and high-school programs are centrally prescribed by the Ministry of Education and mandatory for all private and public, urban and rural schools. Sporadic curriculum reforms amounted to some additions, deletions and reshufflings of program contents. In 1975 a newly elected Liberal Government started a comprehensive school reform, called "The Qualitative Betterment of Education." An ambitious experimental curriculum for grades one to three was drafted and tested in a sample of schools scattered around the nation. In 1978 I was appointed by the National University of Colombia as Advisor to the Ministry of Education for the restructuring of school mathematics. Many evaluations and rewritings of the experimental programs have taken place since the first draft, and a newly revised set of programs for the five-grade

elementary school has been submitted to teachers and universities for evaluation.

About half of the public schools are located in rural areas, and urban public schools are attended only by lower-middle and lower class students. Cultural diversity among different regions of the country, among urban and rural school districts, and among slum dwellers and middle-class families cannot be properly accounted for by a centrally prescribed program, but curriculum differentiation (or even "tracking") is politically unfeasible. A more flexible policy of adapting the programs to local conditions and of encouraging local rewriting of Integrated Units of Learning has been adopted; I find it basically sound, but it makes the evaluation of the experimental testing hard to obtain and harder to interpret.

As expected, more questions than answers have arisen from experimenting with the new grade-school programs. Teacher resistance to change is widespread, and the Teachers' Unions stall reforms as a strategy to add to their clout at the negotiation table. Far-right groups find most reforms "dangerous" or "inspired by world communism," and far-left groups brand them as "useless" or "dictated by U.S. imperialism." Local research in education is minimal, and research made in other countries is often contradictory or easily disqualified. One could quote conflicting findings

about most questions in mathematics education (2).

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(2)
See, for example:
BEGLE, E.G. Critical Variables in Mathematics Education: Findings from a Survey of the Empirical Literature. MAA-NCTM, Washington, D.C., 1979.
FENNEMA, E. (Editor). Mathematics Education Research: Implications for the 80's. NCTM-Reston, 1981.
SUPPES, P. (Editor). The Impact of Research on Education. NAE, Washington, D.C., 1978.

The raging war between "New Math" and "Back to Basics" oversimplifies the issues and. Arguments are heated and emotional, and the facts quoted by each side are easily reinterpreted by the other.

But conspicuously absent from the evaluations, from the political bickering, from the research reports, and from the "New Math" vs. "Back to Basics" polemics is the main issue of this paper: What conflicts arise between universal or standard mathematics and concrete mathematical systems found in the local cultures? How to detect, refine and utilize those culturally transmitted systems for a curriculum reform? What experience with those concrete mathematical systems do children already have before or outside school?

Children are routinely assumed, not only by myopic psychologists but by almost everybody else, to know nothing about real mathematics when they enter school, except perhaps counting up to a certain number. How true is this assumption?

And even far-sighted psychologists and practically everybody else think that school mathematics is a universal subject, whose cultural dependence is limited to the informal language, the mathematical symbols and the trivial examples and story-problems used in the classrooms and textbooks. Is that the important cultural dependence?

3. A Tool for Cultural Research in Mathematics.

It is well known that even illiterate cultures have often quite complex trading systems, astronomical and calendrical calculations, games and sports with complex scoring rules, intricate marriage systems with refined mathematical structures, etc. But it is not so well known that children handle complex mathematical--or at least proto-mathematical--systems in a very skilled way before going to school, or without having gone to school at all.

Many scattered but consistent findings suggested the development of a theoretical tool to handle the empirical data, to lay out the field in a wider perspective, to provide a framework for further research, to shape a disciplined language, and, above all, to enable the mind's eye to have a clear vision of holes. In my opinion, one of the main uses of theory is the possibility of making holes visible. By definition, holes are non-entities, and therefore not directly perceivable. The fact that one does perceive them, points to the existence of an underlying theory, be it only the anticipation of a smooth field of vision. If that much theory exists, and you see edges, you can construct holes. No theory, no holes.

And everybody agrees that the teaching of school

mathematics is full of holes. How to classify and analyze them? How to dismiss pseudo-holes and detect new ones? How to perceive cultural vacuums and pseudo-mathematical emptiness?

The abused fad of the sixties, "the systems approach," by now hammered and chiseled through twenty years of hard criticism, yielded the clues to the understanding of mathematical systems taught in school and of those familiar to children at home and in their neighborhoods. A refined version of a General Systems Theory applicable to mathematical education research is needed. It should cover the whole of existing mathematics without pedantic ambiguity; it should lay out the field and help to indicate holes; ideally, it should overcome the sets-and-logic approach of the New Math, and the numbers-and-routines approach of the Back to Basics people. But it should also meet a seemingly contradictory demand: in spite of the intended generality and universality, it should help detect very fine cultural differences in the proto-mathematical and mathematical systems actually used by children and unschooled adults.

In the late seventies, I attempted to develop an appropriate version of the General Systems Theory in a more general setting (3) and to refine it for the purposes of research and curriculum development in mathematics (4). For the limited purposes of this paper, and for the sake of a minimal agreement in terminology, a short version follows.

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(3) A first draft of the more general version was published by the Colombian equivalent of the N.S.F, "COLCIENCIAS," in its journal Ciencia, Tecnologia y Desarrollo, vol. 4, n.4 (October-December, 1980), 463-482, under the title: "Teoria de Sistemas y Metodologias Cientificas." A fuller account was published by the O.A.S. as an instruction module in the series "Teoria del Cambio," Module 1: "Teoria de Sistemas," in 1982. It is available in Spanish, xeroxed or microfilmed, from the O.A.S. Headquarters in Washington, D.C. or from the Colombian Ministry of Education: Ministerio de Educacion Nacional, Oficina de Documentacion, CAN, Bogota, D.E. Colombia.

(4) The special version for the mathematics curriculum was published by the Ministry of Education as a booklet for teacher instructors in several editions since 1978. It was read as a paper at the Fifth International Conference in Mathematics Education held in Campinas, Brazil, in February, 1979. This paper was later published in the mathematics teachers' magazine edited by the National University of Colombia, Notas de Matematica, n. 10 (1980), 1-14. The title translation is: "The Concept of System as a Key to the Mathematics Curriculum." ["El Concepto de Sistema como Clave del Curriculo de Matematica"].

4. The Internal Splitting of Mathematical Systems.

Let us think of systems in the present context as consisting of components, procedures and relations. More formally, we could think of a system as an ordered triple (C,P,R) consisting of a set of components, a set of procedures and a set of relations.

Components are discernible, multiple, passive with respect to procedures and subject to relations. They are things, objects, elements, points, individuals...to be recognized and handled. Either you see them, or you overlook them.

Procedures are active and point to the practical; they demand time, energy, adroitness. They are actions, transformations, processes, operations, algorithms, routines, functions...to be performed. Either you perform them, or you omit them.

Relations are more subtle and point to the theoretical; they bind things together or separate things from each other. They are bonds or repulsions, similarities or differences; they relate, structure, organize components. Either you get them, or you miss them.

It is instructive to try to define components, procedures or relations. All definitions turn out to be circular. There have been some attempts to reduce procedures to relations and relations to sets of ordered n-tuples. As useful reductions, they might be practical for particular purposes; as definitions, they completely miss the point.

Components can be viewed as the building-blocks of systems. Procedures can be viewed as providing the dynamics of systems. Relations can be viewed as providing the statics, the structures of systems.

Systems have structure, not vice-versa. Several systems may have the same structure. The same components may be rearranged in different structures. You may detect several different structures in the same system; but once you fix one definite structure and forget, neglect, abstract from the others, you start to speak of a definite system with that structure only. You speak about the ordered group structure of the system of integers under addition with the usual ordering, and abstract from the ring structure, or from the partially ordered structure determined by the multiplicative (divisor/multiple) ordering. Here the integers are the components, addition is the procedure, and the usual additive ordering is the relation. But you may have a whole set of procedures, like addition, subtraction, multiplication, Euclidean division, among others; and a whole set of

relations, like strictly less than, less than or equal, strictly greater than, greater than or equal, proper divisor, divisor or equal, proper multiple, multiple or equal, successor, predecessor, opposite (in sign), among others. All procedures mentioned were binary, in the sense that each handles two components at a time to produce others; but there are many procedures that take only one component at a time (unary), or three components at a time (ternary), or n components at a time (n-ary procedures). All relations mentioned were also binary, in the sense that each one holds or does not hold between two components at a time; but there are many relations that hold or do not hold of a single component at a time (unary predicates), or among three components at a time (ternary), or n components at a time (n-ary relations).

Each procedure defines a natural associated relation that holds between the components used as arguments and those produced as results: the raw-stuff-to-finished-product relation. These relations that result (but are different) from the procedures, along with the explicitly given relations, constitute the structure of the system. Relations structure, organize, systematize the system. A system without relations (and hence without procedures) is a "dead" system: it has no structure. A set of components is therefore the corpse of a system. One can define, or mount, or build, many structures on that same set of components,

which is called the universe, or the underlying set of the system.

Structures are abstract with respect to systems. In a sense, structures are that which several similar systems have in common: they may have the same structure in a sense easily made precise. Systems, in turn, however abstract they seem, are concrete with respect to structures. The additive group of integers is a concrete system; the abelian group structure is abstract: many other concrete systems do have the same abelian group structure.

This straightforward way of splitting systems into three very different sorts of entities was most fruitful in the conceptual analysis of every single mathematical system encountered in the mathematics curriculum from pre-school to college. Without the conceptual clarification brought about by this threefold internal splitting of mathematical systems, the cultural analysis would have been impossible (5).

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(5) The power to make holes visible is, as stated above, the main use of theory. The simple theoretical tool of splitting systems as directed enabled me to see the incompleteness of the usual Bourbaki-type of splitting structures into three "mother structures": algebraic, ordinal and topological. First of all, Bourbaki confuses systems and structures; second, algebraic structures are those in which the only relations are those implicitly defined by the procedures; but ordinal and topological structures are just two examples--and not the only ones--of structures in which no explicit procedures are given. They should be called (purely) relational structures. Ordinal structures have only order relations (binary); topological structures have only vicinity relations (neighborhoods can be

thought of as unary relations: V(x) holds if V is an open neighborhood of x; or as binary relations: V(x,y) holds if V is an open neighborhood of both x and y). But there are many other relational structures not classifiable as algebraic, ordinal or topological. For example, a system of points with betweenness and colinearity as (ternary) relations; or a system of straight lines with parallelism and perpendicularity as (binary) relations. They are purely relational, but they are neither ordinal nor topological. Such a gaping hole in the Bourbaki system was invisible without the theory.

5. The External Splitting of Mathematical Systems.

Once internally analyzed into components, procedures and relations, all systems of school mathematics and all systems culturally functioning as mathematical or proto-mathematical outside of schools were seen to exist at three very different levels:

5.1. Superficial level: symbolic systems.

They are the only ones appearing in textbooks. No wonder: textbooks can print symbols only. They are the only ones taught to children. No wonder: teachers can only emit sounds, draw signs, form gestures, and all that is exclusively symbol transmission.

Formalist philosophers of mathematics think--or try to force themselves to think--that symbolic systems are the whole of mathematics. Fortunately they are not. Formal systems are not even the whole of symbolic systems in mathematics: most of written and spoken mathematics consists of informal symbolic systems.

Numbering systems (as different from number systems) are good examples of symbolic systems. Take for instance the Roman, or our own decimal Hindu-Arabic system.

5.2. Ground level: conceptual systems.

Once formalism and behaviorism were assigned their proper niches, constructivist philosophies of

mathematics and cognitive schools of psychology could talk again about constructing, handling and understanding conceptual systems, without guilt feelings. Conceptual systems are those that are assumed by good textbooks, even though they can only talk or write about them through symbolic systems (talking textbooks are now on the market). Conceptual systems are those that are constructed slowly--not always painfully--by children from the earliest months of their lives. Number systems (as different from numbering systems) are good examples of conceptual systems. Take for instance the so-called natural numbers or cardinal numbers with their usual procedures (operations) and relations. "Natural" numbers are of course as "artificial," as much works-of-art, as much construed and constructed, as much man-made or woman-made as anything you name. But they are constructed so early that they seem natural. You may also realize that the conceptual system of natural numbers with their procedures and relations is independent of the numbering system you choose for it, either English-spoken, or Spanish-spoken, or Roman-written, or Hindu-Arabic-written, or binary-coded. It is essential to realize that a single conceptual system at the ground level can appear as a plurality of symbolic systems at the superficial level.

5.3. Archaic level: target systems.

It took a lot of digging to discover the different strata running way underground. Piaget, Vygotsky and Bruner helped a lot. Language analysis and "clinical interviews" of children helped more. But you can do a lot of analyzing and a lot of interviewing without knowing what you are looking for. The key ideas came from an unknown 1973 book by the French historian and philosopher of mathematics, Pierre Raymond. It is unknown to mathematicians and educators because nobody would guess from its title ("Le Passage au Materialisme") that it has anything to do with mathematics; and it is unknown to materialist philosophers because they stop reading it as soon as they find out that it is about mathematics. I happened to like mathematics, among other sciences, and to like materialist philosophy, among other philosophies, and found the book fascinating. It points the way for an archeology of mathematical constructions. It shows how newly constructed conceptual systems in mathematics come from abstraction of active transformations on deeper systems. These are then forgotten. The archaic systems are thus passive with respect to the active system that acts on them. They are concrete with respect to the newly constructed conceptual system, which is abstract by definition: it is abstracted, pulled-out of the concrete system or systems that gave rise to it. They

are natural with respect to the artificial, artfully constructed conceptual system that supersedes or superates the archaic systems. (6).

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(6)
This is a typical case of the dialectical concept of "Aufhebung": to superate in the sense both of making something obsolete and of recuperating the best, the gist, the essence of it. Good examples of target systems are hard to pinpoint, because they are easily forgotten once you have constructed the conceptual system. And you can swear there was nothing deeper than the systems now appearing concrete, passive and natural. It is essential to realize that a single conceptual system can appear as a single construction over a plurality of target systems at the archaic level.

To give an example of target systems giving rise to new systems at the ground level, let us go back to the threefold splitting of systems into symbolic, conceptual and target systems, but now going backwards, or better, forwards, but starting from the bottom up. Let us take up again the example of the natural number system at the ground level with its different numbering systems at the superficial level, and let us try to look for the target systems at the archaic level that give rise to the others.

6. Archaic level: target systems.

Let us think of children playing games with concrete collections of everyday objects. Thinking about collections of pictures, of toys, of silverware or dishware, or better still, about collections of rocks or bottle crown-caps, you may imagine a poor child from a slum in Colombia moving things around, observing, pondering, piling things up, inventing his own games. At first he has some objects to play with, performs some actions on them, and perceives some similarities and differences among them. That is precisely a system with its components, its procedures, its relations. Through grouping the objects by hand, by visual-neural "Gestalten," by common properties, by comparisons, his original system of rocks and bottle crown-caps becomes the target system for a new system of collections of such objects. The new objects are now collections,

sub-collections of the original one (usually not including the whole original collection or the empty collection). Then he begins to assemble and disassemble collections out of other collections; he groups them and splits them; he sets aside the crown-caps of a given brand of soft drink from the rest; he moves a whole pile or a train of crown-caps onto or next to another. He perceives some similarities and differences among the collections. That is precisely a new system: its components are the sub-collections; its procedures are reunion and complementation; its relations are inclusion, disjunction, not having anything in common with, having a larger number of objects than, having a smaller number of objects than, having the same number of objects as...

As the child plays with the original target system, he begins to construct a very interesting conceptual system, described by Piaget in detail, and classified by him as a "Groupement": almost a group, but not quite; almost a lattice, but not quite. Mathematicians may not like this hybrid monster, but it is enough to play with crown-caps with your neighborhood pre-schoolers for a half-hour to convince yourself that the conceptual system they are constructing is not the usual Boolean algebra of sets with union, intersection and complementation, nor the usual Boolean lattice of sets with inclusion New-Math-fanatics think they are. But it is not the natural number system

Back-to-Basics fanatics think they are constructing either. My hypothesis is that the failure of the New Math is due to the dissimilarity of the conceptual system of teachers and textbook writers (the usual Boolean algebra of sets or its equivalent lattice), and the culturally constructed system of children: the system of collection games, which has, among others, a "Groupement"--structure. But now, if you noticed the last three relations perceived by the child in his collection game, they seem to have more to do with number: less than, greater than, equal. But not yet. My hypothesis about the failure of "Back to Basics" is that it tries to force numbers down the throats of children through symbolic routines (the words one, two, three... are also symbolic), before they finish enjoying the culturally (for their age and environment) more appropriate collection games.

Soon enough they realize that those relations they may express (at the symbolic level) as "smaller than," "bigger than," "the same as," naturally split the sub-collections into batches having the-same-number-of crown-caps. This equivalence relation is built out of the two other order relations "smaller than" and "bigger than" when the ordering fails. The dependence of many equivalence relations on order relations is not found in mathematical textbooks. But it is simply detected as a failure to order: ask a group of children to queue up in order of height; if you see two children pushing each other too long, they are the same

height.

Notice that these relations we have expressed at the symbolic level as having-less-elements-than, having-more-elements-than, and having-the-same-number-of-elements-as, do not depend on the concept of number. On the contrary: the concept of number is built out of those three relations, when the game of sub-collections is taken as the target system for a new conceptual system: the natural numbers starting with "one," with "going up one" as the natural procedure, and "being next to" as the natural relation. The procedure is unary, not binary like addition. Peano was right in more ways than he thought, when he formalized the natural numbers in his first version, starting with one, and using the successor as the only operation. Children like his system.

But they do not like the natural number system of teachers and textbooks with its four operations (none of them the one they like) and their two relations of additive order: less than or equal, greater than or equal. Children accept rather the relations: strictly less than, strictly more than. But for mathematicians these relations are not reflexive, and hence, by definition of order relation, they should not be called "orders." Calling them "strict orders" is a contradiction in terms, widespread usage notwithstanding. At first, children build a conceptual

number system for each type of collection game: two-bottle-caps, two-rocks, two-candy... The fleeting but real existence of these concrete number systems is widely attested to by linguistics and cognitive psychology. They are intermediate stages in the construction of the natural number system, abstract with respect to concrete games of "smaller," "larger" or "equal" sub-collections, games which in their turn were abstract with respect to games with the crown-caps, rocks, pictures or pieces of candy themselves.

Old-style arithmetics still talk about "concrete numbers" like two caps, and "abstract numbers" like two. This distinction is worth reviving. It marks the culturally relevant proto-mathematical systems out of which the mathematical system of the natural numbers with their operations and relations is constructed at the conceptual level. It is the one Norma X knew how to handle.

Once the natural number system is constructed as abstract, and the child has played with it for years, counting forward and backward, jumping two or three numbers at a time, he begins to notice that going-up-two, or going-down-two, are new and interesting procedures in themselves. Now he begins to take the once abstract natural number system as target system; the procedures that acted on the natural numbers moving them up or down begin to take up a personality of their own. He is beginning to construct the

signed integers. Then he sees that the natural procedure between two signed integers is to apply them successively to a natural number. New procedures and relations begin to be constructed conceptually, and a new system, abstract with respect to the target system of natural numbers, begins to take shape. What a mathematician would call the composition of two operators is for the child the natural addition of integers: if you go up four and then go back three, it is the same as going up one. Is that addition or composition? The child couldn't care less: he can perform the new procedure on the newly formed components of his conceptual system. Integers are natural when natural numbers become so familiar as to seem natural. But in the usual textbooks, all that is found about this culturally natural construction of the signed integers is a confused and complex double-talk about the integers being equivalence classes of ordered pairs of natural numbers. Only very fine archeological work could detect that those equivalence classes are precisely the dead bodies (graphs) of the live and active signed integers as up-and-down operators children love to construct. In fact, the active operator +2 applied to the natural numbers as target system has as its graph the set of ordered pairs (0,1), (1,3), (2,4), ..., (n, n+2),... But using that as a definition for a child who can construct the active operator by him- or herself, is as awkward as defining one hour to be the set of ordered pairs (midnight, one a.m.), (one a.m., two a.m.), ..., (eleven a.m., noon), (noon, one p.m.), ...

(eleven p.m., midnight). Would you like that? And what about the hour from half-past one to thirty-to-three? Any honest youngster will tell you he doesn't like defining integers as equivalence classes. And you cannot list them all anyway, either. But you can certainly build the concept as soon as you can count up and down the natural numbers by ones or jumps. And those are hard procedures to master. Back-to-basics fanatics are right on insisting to New-Math fanatics that they should not teach integers as equivalence classes; but they are wrong on trying to force children to master the integers through symbolic routines. They will be forgotten and confused right after the final exam if they are not articulated onto the up-and-down jumping procedures.

By the way, the infinity of natural numbers and positive integers (as operators they are not the same as their targets) is easily understood by young children as endlessness (in-fini-ty) once they master the going-up procedure. Let me finish this section with the following true story:

In 1978, Santiago Z, then a second-grader, started a contest with me: he would name a big number and I would name a bigger one, and so forth. He started easy: "Ninety!". I said: "Hundred ninety!". "Two hundred!" he said. "Two thousand!" I said. "Ninety thousand!" was his bid, and I retorted: "Hundred thousand!". He became exasperated and

shouted: "Infinity thousand!". I lost the game.

7. Ground level: conceptual systems.

It should by now become clear that the ground level is shifting upwards as soon as a system is mastered and can become the target for the next construction. It is to Piaget's credit that he noticed that the new construction results as a "reflexive abstraction" from procedures, not from objects or relations. Procedures acquire the status of objects when the archaic system is forgotten. This happens with the signed integers and with the rational numbers, which arise culturally as procedures on target systems, and then become passive objects, representable as points on a static line. Teachers and textbook writers think those are the integers or the fractions, and confuse the issues hopelessly for the youngsters that are trying to build the active procedures and master them. The first conceptual system which is mathematical with respect to the proto-mathematical target system of the crown-caps was the "Groupement" of crown-cap sub-collections. Out of many such collection games arises a second conceptual system, that of the natural numbers starting with one and having as natural procedure going up to the next. Out of this initial conceptual system the full system of natural numbers with many operations and relations can be constructed at the conceptual level. The

original conceptual systems of sub-collections are superated, forgotten. The procedures of going up or down by jumps give rise to a new conceptual system of signed integers with culturally natural procedures and relations. Once this basic system is built, it can be enlarged by introducing new procedures and noticing new relations.

Then, this new conceptual system takes up a life of its own and nobody notices its archaic roots any more. The "victims" of the going-up or -down procedures, that is, the target systems, are buried deep, and the conceptual system of signed integers with their usual operations and relations is ready to become the target system for a new conceptual system: the integer-valued functions on the signed integers cherished by number theorists.

To divert attention from symbolic systems, I have tried to avoid all mathematical symbols in the preceding text. To stress the point I am trying to make, it must be said that the words I have used to describe the target systems and the conceptual systems are not really the objects, procedures or relations: they merely refer to them; but the use of everyday language was meant to convey the concept of archaic and ground-level systems as different from the symbolic systems. I am sorry, but I cannot possibly print here a target system or a conceptual system; I can only give you some hints on how to construct a conceptual system once you

have identified a familiar target system. I wish I could spare you the construction work (direct telepathy maybe?). But then you might also miss all the fun.

8. Superficial level: symbolic systems.

Now, when it comes to symbols, it is just child's play to devise a symbolic system once they master the conceptual system. Try to design a symbol system for the more primitive system of crown-caps. Or for the system of collection games. Or a system for the natural numbers, or for the integers, or for the rational numbers which is different from the run-of-the-mill numbering systems.

Your design might not be very good for certain things, like adding or multiplying large numbers; but it might be very good for other things. A simple tally of equally spaced bars is very nice for the newly formed conceptual system of natural numbers just pulled out of the target systems of the collection games; you do not need a symbol for the natural operation of "going up to the next": you simply write one more bar to the right. And if you are careful, the order relations are easily checked by length comparison: ///////

//////////.

If you get tired of counting bars one by one, try the usual trick of crossing out fives with the five-bar: 𝍸 is a

very nice symbol for the number five.

Different cultures have developed different symbolic systems for the same conceptual system of natural numbers. How do we know it is the same? The "sameness" of the conceptual system can of course only be inferred out of coincidences, easiness of translation, success of trading practices, or other observable symptoms. It is clear that at this basic level we have called the ground level, a large degree of universality is attained. But is it only at the symbolic level where the cultural diversity is located?

My answer is a definite "No!". The cultural diversity is also located at the archaic level of target systems, and that location is more important and far deeper than the superficial diversity of symbolic systems.

9. Pedagogical Consequences.

The first general idea is that the teacher, the curriculum designer and the researcher should try to start from the bottom up: exploring the diversity of culturally concrete target systems that give rise to a conceptual system.

In the present theoretical framework, mathematics at the

conceptual level cannot really be taught. One can conform the right environment and give the right cues to promote meaningful play with the culturally familiar target systems, to help children construct the conceptual system; once they have constructed it, they should be encouraged to invent their own symbolic systems.

There will be a plurality of target systems that may give rise to the same conceptual system; the choice of those with which the children will be encouraged to play has to be made in terms of cultural familiarity and usefulness. There will be a plurality of symbolic systems that arise out of the same conceptual system; the choice of those the children will be encouraged to develop and master has to be made also in terms of cultural familiarity and usefulness; but if a child, or better still, a group of children, develop their own system, they should be encouraged to play with it and to compare it with the culturally more widespread systems. If they understand the conceptual system, they will easily make the translation; if they do not, they will not even know it is a translation.

A second general idea is a counterpoint to the first: it states that starting from the symbolic systems down is counter-productive. The mechanical mastery of symbolic routines blocks the concept-building process. Symbols can and should be used to help the conceptual construction along

the way, and it can be argued that the concept cannot be fully developed without some form of accompanying symbolism. But the priorities should be clear.

If the children know some symbols, one can start down to the possible underlying concepts and the certainly underlying target systems. If they progress up from target systems to some concepts, one can start seeking for symbols. One should foster these ups and downs, but remarks that the main trend has to be upwards, starting from different target systems that have the same structure. Once the conceptual system is partially built, and a particular symbol system is attached to it, try different symbolic systems that have the same meaning. Try to infer whether the meaning is really there: go all the way down to different target systems to see if the concepts are correctly applied to them. Then go up the ladder again.

In the book mentioned before [Section 5.3], Pierre Raymond shows how this simple pattern of using one or several target systems as concrete raw materials for the construction of abstract conceptual systems is repeated throughout the history of mathematics. My suggestion is to let "ontogenesis," the re-construction of those systems by children, follow at least some of the paths of "phylogenesis," the construction of those systems in the history of mathematics.

But I am not stressing the usual (and wrong) platitude of teaching mathematics only through so-called concrete methods or concrete materials. Raymond observed the above-mentioned relativity of the classification of a particular system as concrete or abstract also at the historical level. Piaget observed it at the genetic level. But the pattern is the same: a concrete system is acted upon by an abstract conceptual system; once a good symbolic system is developed for the active system and enough familiarity is achieved, a new active system uses the former as target system, and the new conceptual system can be so successful that it now becomes concrete and the first concrete system disappears into the background and is easily forgotten.

I am advocating a rescue operation to recover those concrete systems, where the original cultural dependence will be found to lie. They might turn out to be very abstract themselves, but that does not matter. The word "concrete" is very relative indeed.

Piaget has sharply criticized the confusion between these activities in school and the truly active methods. Any perception, any meaningful word of a three-year-old is already abstract: it lumps together many objects and forgets, or abstracts from, many otherwise important differences. It takes a lot of mental activity to construct

that. It is the mental activity on the currently concrete systems that has to be stressed.

Vygotsky remarked also that teaching methods used for mentally retarded children, based on concrete, look-and-do techniques, had resulted in profound disillusionment:

> It turned out that a teaching system based solely on concreteness--one that eliminated from teaching everything associated with abstract thinking--not only failed to help retarded children overcome their innate handicaps but also reinforced their handicaps by accustoming children exclusively to concrete thinking. Concreteness is now seen as necessary and unavoidable only as a stepping stone for developing abstract thinking--as a means, not as an end in itself (7)

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(7) Quoted with permission from VYGOTSKY, L.S. Mind in Society. The Development of Higher Psychological Processes. Harvard University Press, Cambridge -London, 1978, p. 89.

The above ideas forced a thorough rethinking of all systems taught at the grade-school and high-school levels, from the natural numbers with their usual operations and relations, to the real functions of elementary calculus with their operations and relations. Many surprises were found at all levels, from the most concrete collections of familiar objects children use as target systems to build their counting numbers, all the way up to the bizarre behavior of functions and derivatives.

A full account of the research done on all those systems will be the subject of a forthcoming book. Many aspects of that research could be called cultural, anthropological, linguistic, psychologic or mathematical. But the general pattern repeated itself over and over. Seemingly trivial details of language, learning troubles, apparent mistakes, negative reactions of children, led the way to unearthing the culturally relevant systems, pinpointing the conceptual system and relating the different symbolic systems attached to it. A few short examples follow in Sections 10 to 15.

No contention is made about the representativity or generalizability of these examples. It would be hard to separate what is specific to a particular child, a particular sub-culture or a particular country, from what is general or universal, or to pinpoint what is due to cultural factors, or to genetic factors, or to an interaction of both. It is

clear, for instance, that the peculiar objects or local toys children play with are not the same in different cultures; they are not the same even in different strata of the population of the same city of Bogota. But why have childrens' games turned out to be more universal than was imagined? Why do developmental psychologists trace strikingly familiar patterns of growth for children from Geneva, Paris, New York or Jakarta? The hypothesis underlying the presentation of examples from my limited experience is that the action schemes of children will be found to be very similar in different cultures, no matter how different the targets of those actions or the linguistic labels attached to them may be.

If mathematics arises from the articulation of those action schemes, as Piaget contends, and not from the reactions of the target objects or the repetition of spoken or written symbols, the following pattern of a correct pedagogy of mathematics can be transferred to wholly different cultures and environments: explore first the mathematical and proto-mathematical activities of the local culture; pick up the language actually used by children and adults to represent the corresponding objects, procedures and relations; start from the concrete target systems found in that culture and use the local language to elicit those action schemes or activity patterns on the target systems. Watch for the symptoms of the actual construction of

The theoretical key was Piaget's distinction between collections (both figurative and non-figurative) and operational classes. Concrete figurative collections have to be visible as one simultaneous "Gestalt," and they must have at least two elements. Concrete non-figurative collections must also count at least two elements, and all members of the collection must have a common property and fall under the same label. There are no empty collections, or unitary collections either. These collections are not mathematical sets: they are systems with many inner relations and active operations and procedures. Children never perceive members of their families or the buddies of their class by children as sets of loose, unordered, unstructured elements. The set with its dead elements, its frozen procedures and departed relations is just the corpse of a system. No wonder children do not like sets; but they love systems. Where in the world is the child that accepts that the set [a,b,c] is the same as the set [c,b,a]? Such equality is possible only at a high level of abstraction, when all the "oblivious functors" have deleted all traces of structure in the original systems and have dumped them into the obscure depths of the category of sets.

I am not denying that it may be interesting, and even necessary, to study corpses of systems at later times. In biology it is also necessary to dissect dead frogs and to kill cells with dyes to see them through the microscope. I

oncceptual systems, and let several symbolic systems flow or attach themselves to those conceptual systems. My contention is that this pattern of educational activity is generalizable to the teaching and learning of mathematics in other cultures.

10. Set Theory.

Set theory as conveyed by New Math textbooks turned out to be very alien to our cultural environment. Of course, British and Dutch school teachers found that out very soon after the introduction of the New Math in their own cultural settings, and Japanese teachers rejected it out of hand after a few months.

Nevertheless, a little more digging showed deeper strata that were easily and playfully handled by Colombian children. Many culturally familiar situations that give rise to the counting numbers were found to be natural and interesting for children. I call them "collection games." Children readily identify, use, count, label, assemble and partition collections of everyday objects in and out of school: their marbles, their toys, their school utensils, furniture, silverware and dishware, postage stamps, picture cards, you name it.

am only saying that children do prefer live frogs and wiggling cells. They also prefer live, bound and structured systems, to dead, loose, unstructured sets. If the words "set" and "element" are not forced on the child and one abstains from empty and unitary sets, the child is home free in that part of set theory that reflects concrete collection games. One is reminded that for the Greeks, and all the way down to Renaissance arithmetics, zero and one were not numbers: numbers started at two. We still agree with our ancestors that unity is neither prime nor composite; but we balk at the perfectly parallel statement that unity is neither even nor odd. But they accepted both, simply because unity was not a number.

The culturally natural procedures with collections were found to be union and complementation. Children refused to use the connective "or" in connection with union of collections, as all set-theory textbooks prescribe: they prefer the connective "and," which textbooks say has to be used for set intersection. But children insist on saying "The blue blocks and the disks" for the union, and use no connective for the intersection: "The blue disks." Children were right. So much the worse for textbooks.

Such simple explorations of the cultural world of children's collection games, carried out in the mood of a mathematically trained cultural anthropologist, allowed me to

rewrite much New Math material that teachers used to repeat without much understanding, and children used to repeat without any understanding at all. Many virtues of the proper version of set theory were recovered, and at their root many well-known vices were detected.

11. Logic.

Another surprise was the contrast between the "New Math Logic" now taught in many grade schools, and the culturally specific logical language games played by children. It turned out that propositional logic is not understood by young children as textbooks assume children often have a pure reward-punishment idea of telling the truth. But an appropriate version of the so-called deontic logic of some wayward logicians was found out to be perfectly understood by youngsters in a very precise way: they are constantly given instructions, orders, prohibitions, promises of conditioned rewards and punishments. But in this culturally tuned setting the usual tautologies are no longer valid. The connective "and" is not commutative; try to commute: "Take off your shoes and pull out your socks." Because of the time-dependence of conditionals, the counter-reciprocal is not equivalent to the original sentence, and "If p, then q" is interpreted by children as "either p or not q"; textbooks assure you that such transformation is incorrect, and that

the right one is "either not p or q," which children refuse to accept. In fact, "If you behave, you may go to the movies" is always interpreted by the child as "Either you behave, or you may not go to the movies," and never as "Either you do not behave, or you may go to the movies," which is preposterous. Children are again right and textbooks wrong. Again I say: so much the worse for textbooks (8).

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(8)
A preliminary study of time-dependent connectives and their use in ordinary language was recently published in Spanish in: VASCO, Carlos E. "Conectivas Secuenciales y la Formalizacion del Lenguaje Ordinario." Matematica-Ensenanza Universitaria (Bogota), n.27 (June, 1983), 12-23.

12. Natural Numbers.

Natural numbers are associated with concrete systems of collection games, with number-words attached as quasi-adjectives to plural nouns, or with rote-memory number lists attached to objects of a concrete collection through finger-pointing. Dice, dominoes or playing cards may be well known by very young children, who can tell the patterns as "Gestalten," without counting dots or spades. The words they use may collide with the decimal numbering system; for instance, "ten" is a shorter word than "nine," but "10" has more symbols than "9." The numbers 11 and 12, when read aloud, have nothing to do with the symbols; 13 and 15 are not quite read "threeteen" and "fiveteen" as they should be. In Spanish, the numbers 11 through 15 present a similar problem. But we do not have the problem the Germans have: 21 is read in reverse as "einundzwanzig"; compare with the Latin "undevinginti," which stands for our 19. And let us not even tackle the French problem of "quatre-vingt-dix-neuf"...

To mention only one operation, a study of division guided by a few hints given by the Italian mathematician Emma Castelnuovo, led to the detection of two different types of culturally familiar division processes, none of which agreed with the one taught in school:

12.1. To give the same number of items from a given batch of

candy to each one of the children in a given group. The result is a number of items.

12.2. To split a fixed amount of items into batches of a given size, each batch to a child. The result is a number of children.

Children in Colombia speak of the first process as "dividing candy among children" ["Repartir entre"]. The expression used for the second process is rather difficult to translate into English: they speak of dividing at-a-time ["Repartir de a"]. But both processes can be carried out without using multiplication, and therefore do not coincide with the division process taught in school, or with the words used by textbooks: "Divide by" ["Dividir por"]. The independence of the culturally familiar division processes from multiplication is easily observed outside the schools, in spite of the textbook identification of division as an inverse operation to multiplication. I was told by a French mathematician that there are native groups who practice addition, subtraction and division, but do not know how to multiply. If confirmed, their existence would support phylogenetically the ontogenetically observed independence of division from multiplication.

13. Integers.

Children were found to reject instinctively the negative integers. These constructs have been known for centuries, but they were not easily welcomed into mathematics. The master of Renaissance algebraists, Cardano, called negative solutions to quadratics and cubics "fictitious," "useless," and even "false" (9). We are never quite convinced that minus times minus gives plus; we decide to learn that by heart as a rote rule, without trying to understand.

But children can construct the positive and negative integers as active conceptual operators that move things around: natural counting numbers, streets along an avenue, steps in a staircase, and other things more familiar than debts or temperatures below zero. The integers are active operators that move things up n steps (+n) or down n steps (-n). The identity operator does not move things at all (0). But then, the natural procedure between two of these active

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(9) CARDANO, Girolamo. The Great Art or the Rules of Algebra. Translated and edited by T.R. Witmer. The M.I.T. Press, Cambridge, MA, 1968. See pp.9, 11, 29.

operators is not addition: it is the successive application (or composition) of the two operators: we had already remarked in Section 6 that going up four (+4) and then going down three (-3), is the same as going up one (+1), that is: $+4 \circ -3 = +1$, where the small circle represents composition of operators, just a fancy and culturally foreign way of saying that you apply one after the other.

From this viewpoint it can be seen that debts or temperatures below zero are not the right models for the negative integers. They are rather two more target systems for the culturally familiar operators that act on the natural counting numbers, the steps in a staircase or the streets along an avenue. Indeed, walking n blocks North or South changes the numbering of the side streets that go East-West in a definite way: walking four blocks North and then three blocks South is the same as walking one block North. But I haven't told you what street we are at. The target system is one thing, and the active conceptual system is another thing.

The cultural dependence of the target system was evident in an experimental trial of this system in a slum school in Bogota. In this city, the North is a fashionable suburb, while the poor live in the South Side. The obvious symbolic system for the integers in this system is 4N, 3S, 1N, etc. Identifying 4 and 4N by dropping the "N", and writing 4S for

negative-four struck the slum kids as an insult: "Why is it taken for granted that "4" means "4 North"?". "Why are Northern numbers called positive?". "What is so negative about living in the South Side?". The kids were right, and we had to change the system to East and West, because there are very few more Avenues East of First Avenue, and nobody would feel insulted.

Finally, for those operators there is a natural change-of-orientation operator, called "minus" and written "-". It is obvious that if you change orientations twice, you leave them as they were before. There is no rote "rule of signs": just the straight-forward understanding of what happens if you reverse orientations twice. But notice that the "minus" operator is now an active monster that eats integers. Functions are beginning to appear as active operators that act on the now familiar integers, which will in turn fade into a target system. The "minus" is a unary monster that eats integers and spits them back reversed. If it bites you twice, you end up unharmed. Some boys in the State of Antioquia recalled hearing their grandfather tell this story about witches: if you cut a witch once with a knife, she will come back at you and cut herself once more, to cure the first wound with the second. Do you think those boys from Antioquia will ever forget what happens if you apply "minus" twice?

14. Fractions.

This system is the headache of all math teachers. College freshmen seem to have forgotten all about fractions, as any teacher that has corrected tests about derivatives of fractional powers or integration of rational functions knows well. The proposed remedy of the Back-to-Basics movement is to drill much more on the fractions. It has been tried, but the number of rote exercises seems to affect little or not at all the difficulties with fractions in the years ahead; what those exercises are known to affect is the attitude towards mathematics: more kids hate math, and more adults fear fractions.

I began to think fractions were too hard for third-graders. As I pondered moving them up to the fifth grade, I saw a group of first graders in a Gym class following without any trouble a series of instructions about half-turns and quarter-turns. There was a culturally friendly target system for fractions!

Very young children know a lot about half-gallons, quarters, pints and half-pints, as long as it is milk. Luis Eduardo Y and his fellow mechanics know a lot about fractions of an inch. Girls know a lot about fractional lengths, as long as it is string or ribbon. In metric countries

fractional lengths are even easier to handle: no jumps from twelve to a foot, three to a yard or God-knows-how-many to a rod or mile.

But the same child that knows how much is it when you pour a half-gallon of milk into a container, and then a quarter-gallon more, will be totally baffled before the hieroglyph "1/2 + 1/4". Should I add? multiply? cross-multiply? find the L.C.D.? We must have jumped too fast to the symbolic system without any construction of the pertinent conceptual system.

"Fraction" is an ambiguous word: it refers to rational numbers and to fractional symbols. Is it 1/2 the same as 2/4? Obviously not! The first numerator is odd; the second is even. But somehow, they are supposed to be the same. The same what? Well, if the child has not built the right conceptual rational number, the question cannot be solved at the symbolic level. But at the conceptual level, rational numbers have no numerators or denominators. They are active monsters. Once you know the halving monster, you don't care if the teacher writes 1/2, 2/4, .5 or even 50%. It is all the same. But now we know the same what: they are just funny disguises for the same halving monster.

For children from six to twelve (at least) it is very important to know what the monsters are supposed to eat.

They eat turns, they eat lengths of rope, they eat certain natural numbers, like tens or dozens. They even eat quantities of milk or areas of concrete surfaces. I have studied all these target systems, and it is clear that the hardest is the one about areas. But it is the easiest to draw on the blackboard or on the textbook, and it is by far the commonest; but it is the culturally wrong target system to start from at least here in Colombia.

It is also clear that the usual words for transforming fractions, "reducing," "simplifying" and "amplifying" are culturally confusing; it is said that to simplify is the same as to reduce a fraction, and that the contrary is to amplify the fraction. But children associate a reduction with making smaller, and an amplifier with making sounds louder. So I propose to use "simplifying" and "complicating," meaning just to make more simple or more complicated at the purely symbolic level. At the conceptual level, rational monsters really enlarge and shorten what they eat (except of course the tame monster, who does no harm to his victims, and he is "Number One"). It is important to realize that these monsters do not eat pies or sheets of paper, as many teachers seem to think. They eat the mass or volume of pie dough or the area of the sheet. "Half-pies" are quickly identified as phoney: if you will allow me to eat my half-pie, I'll choose the upper half, because it has all the frosting on it. Or if you allow me to choose my half-an-orange, I'll choose the

inner half and leave you the rind. If you ask a child what do you get if you split a sheet of paper along the middle, he won't answer "Two half-sheets" (unless he knows you want that answer). He will answer: "Two little sheets of paper," and he is right. The halving monster doesn't eat the objects themselves: just their areas, lengths, masses, weights or volumes.

Again, the natural procedure to apply to two monsters at the conceptual level is not addition: now it is again composition, or better still, applying them one after the other. Children learn fast that one-half of a half-gallon is a quarter; that one half of a half-inch is a quarter of an inch, etc. But they do not know what is it if they see something like: "(1/2) x (1/2)." Why the multiplication symbol? Why read "by," or "times," when all there is to it is a simple "of"? When the children get bored with the target systems because they have already constructed the rational monsters, they say simply "a half of a half is a quarter." If you ask them for the target systems, they shrug you off: "It's the same." When a child says "It's the same," he has constructed a conceptual system: he has learned mathematics.

15. Geometry.

Geometric systems were also packed with surprises. I must start by confessing that the New Math way of stating that line segments are sets of points used to leave me cold, or worse: frozen. There are too many sets of points that are not lines, and the dogmatic tone of infallibility assumed by the teacher who repeats the dictum is simply infuriating. When one day I saw a child looking at a line segment with a magnifying glass to try to see the points, I had enough of the New Math geometry.

Why not just play with line segments as primitive elements in a delightful system where points are unknown? Rethinking systems of segments, it was clear that there were many interesting operations or procedures to perform on them: sliding, turning, aligning them end to end, doubling them, bisecting them and what have you. And a host of relations can be found: parallelism, incidence, congruence, perpendicularity.

I started playing segment games with children. Children in the slum areas found it hard to say in Spanish "segmento"; they said "semento," which sounds exactly like "cemento" ["cement"]. Thus we decided to call them simply "rayitas" ["little strokes"]. The stroke games were simple but active; children were eager to move the sticks that stood for segments or strokes all over the place, turn them, join them, slide them. Questions about relations would stop them cold.

One could almost see the neurons firing hypotheses; usually no words were uttered; maybe a quick sliding of a lollipop stem or a length check with thumb and index as calipers. One could see the eyes gleam at the flash of an insight. And a smile. All my philosophical training started to bubble in my mind: is this the long lost relation between theory and practice?

Indeed, procedures and operations reflect practice; relations reflect theory. Procedures are active; they have a priority in practice (you have to do something to perform them), and they have also a priority in theory (you have to do something to check a relation).

The priority of action is well documented by Piaget. He defines operations as internalized, reversible and coordinated actions. Vygotsky speaks of "the internal reconstruction of an external operation" (10).

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(10) Op. cit., p. 56

Learning is therefore the performance of actions, operations, procedures and algorithms (at the symbolic level, procedures are called algorithms). Teaching is therefore the facilitation of such performance.

These observations help one to appreciate functions as active transformations, not as relations, and much less, as sets of ordered couples. And they help one to rediscover active geometry.

Once, I drew a line segment downwards, and then labelled it A to B, starting at the point closest to the chalk:

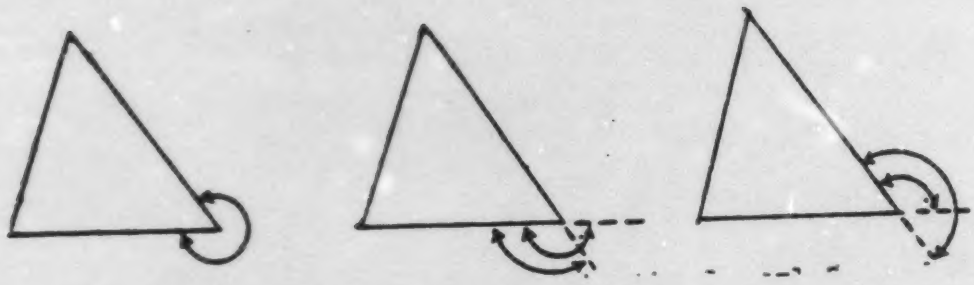
I said: "Let this be the stroke AB". One youngster objected: "You drew it downwards! Shouldn't it be the stroke BA?" I was caught off-base. We forget the motion, the action; children do not. For us a segment is a segment, dead, cold, frozen. It is indifferent if it is called AB or BA. For children, it is still alive, warm, directed downwards. I tried to imagine how many teachers have

answered the same question with a stern face and a dogmatic "ex cathedra": "It is the same. Period." The same what? The same for whom?

The same group of children in the Gym class following instructions like "A half-turn to the right!", who led me to the earliest target system for fractions, solved also the problem about angles. There it was, the long lost active angle-of-turn, right at the first grade. Children's geometry being dynamic, the right concept of angle is not the pair of half-lines (rays) joined by one end. It is the amplitude of a turn, and it has from the very beginning a definite orientation, an inside and an outside. But the angle drawn on the blackboard is already dead. It has lost the orientation, the inside and the outside.

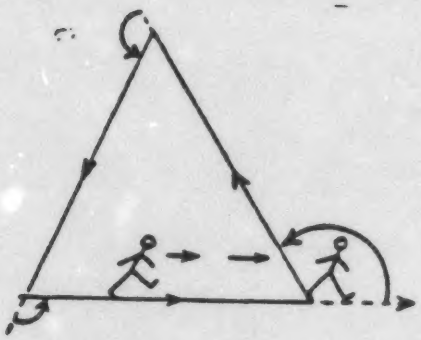
In 1978 I started redesigning the geometry programs from first to ninth grade, reconstructing a whole array of geometric systems I called "active geometry." It starts acting on the child's own body and his toys as target systems. A conceptual system of transformations is constructed on those systems. And many possible symbolic systems can arise from the conceptual system.

Just an example. Walking along line segments and turning with your own body at corner angles solved the eternal query of students about external angles of triangles.



Question: Which one is the external angle?

If you look at the dead figure, you are lost. No definite answer is possible. But just chalk down a big triangle on the floor, and walk around it. There is no doubt about the meaning of the external angle you have to sweep with your front foot to regain the next side of the triangle after you overstep the vertex. The obvious fact that, when you walk around the whole triangle, you end up facing in the same direction you started, gives dynamic, active, live meaning to the fact that the sum of the external angles is a full turn (360°).



Answer: Just walk around the triangle!

But then, what is the meaning of the statement about the sum of the internal angles being 180° (that is, a half-turn)? Just try to walk around the triangle sweeping the internal angles with the foot that drags behind. Much to everybody's surprise, you end up facing backwards. Just try it. Don't miss the fun of active geometry!

Seymour Papert had also discovered this type of dynamic geometry around the same time. In his 1980 book "Mindstorms," he talks about children mimicking the LOGO-Turtle with their own bodies, to be able to program a circular trajectory. He calls this "body geometry" and "syntonic learning":

> The Turtle circle is body syntonic in that the circle is firmly related to children's sense and knowledge about their own bodies (11).

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(11)
PAPERT, Seymour. Mindstorms. Children, Computers and Powerful Ideas. Basic Books, Inc. New York, N.Y. 1980. Quotations from pp. 58 and 63 (with editor's permission).

But to do active geometry one does not need a sophisticated computer or a well-implemented, interactive LOGO-language; My assumption is that you need only a sensitive ear to listen to unsophisticated children talking in their own underdeveloped language, and a sensitive eye to discover the culturally relevant target system. Then you can start developing active geometry, not only at M.I.T.'s Mathland environment, but in every grade school in the country.

Again, in Papert's language, I am advocating "cultural syntonicity." He illustrates this concept by relating angles to navigation by means of the LOGO-Turtle:

> One of the most widespread representations of the idea of angle in the lives of contemporary Americans is in navigation. Many millions navigate boats or airplanes [...] The Turtle connects the idea of angle to navigation, activity firmly and positively rooted in the extraschool culture of many children. And as computers continue to spread into the world, the cultural syntonicity of Turtle geometry will become more and more powerful (12).

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(12) Op. cit., p. 63

I cannot judge how firmly and positively rooted navigation is in America; in Colombia I find it too esoteric: none of the children I know in the public schools of Bogota have this experience. Perhaps in the Caribbean or Pacific Coast of our country there might be some children who navigate boats; but none of them will have any chance of interacting with the computer Turtles of the LOGO-environment, at least not before the 21st century. I do not think we should wait that long: I am advocating cultural syntonicity, not only of geometry, but of every mathematical system; and not only in a LOGO-environment, but at any school, however remote from yachts or computers. Let us start connecting any and every mathematical idea or system to an activity firmly and positively rooted in the extraschool culture.

16. Conclusions.

This anthropological voyage into the uncharted regions of the Colombian grade schools changed my mind about learning mathematics: it revealed that making mathematics one's own can be as culturally dependent as learning literature or history, and not only at the linguistic or symbolic level. The apparent a-historicity and a-culturality of mathematics is a fiction of an adult mind that has reached the plateau of formal thought and has repressed the historically conditioned, culturally dependent reasoning of all children

and most adults, at least when those adults are not trying to "think mathematically;" that is, practically all the time.

The problem is that the math teacher is one of the few adults who is trying to "think mathematically;" and the writer of math textbooks is even worse. The main obstacle the Colombian experimental reform has found is certainly not at the level of children: they love it. It is at the level of teachers, especially at the level of teacher instructors and supervisors. They know all the answers. They are sure mathematics is the universal language. They think they can evaluate objectively what children know about mathematics. I think they just evaluate what children remember about those disembodied symbolic systems most teachers think are mathematics.

A revered teacher and researcher of mathematics, Edward G. Begle, thought to the last minute of his life that mathematical systems were formal, symbolic systems. His survey of math education research hit on the notion of mathematical systems, but the prevailing formalist philosophy of mathematics misled him to identify mathematical systems at the conceptual level with formal symbolic systems, and, of course, to forget target systems completely. (13).

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(13) See BEGLE, E.G. Critical Variables in Mathematics Education: Findings from a Survey of the Empirical Literature. Mathematical Association of America--National Council of Teachers of Mathematics. Washington, D.C., 1979.

Begle died in 1978 at the age of 63. In the introduction to his book (pp.1-3) he studies the basic notion of mathematical system. But he reduces mathematical systems to our superficial level: symbolic systems.

Most math education research starts from the same confusion and ignores target systems altogether. My prediction is that such research will progress little or nothing from now on.

Perhaps alternative teaching methods might accelerate learning of symbolic procedures; but the researcher will not know why. New methods based on Artificial Intelligence models will come to the fore; but up to now machines can perform only symbolic algorithms. Conceptual understanding, transference to new situations and production of new mathematical concepts from target systems will remain in the shadow.

As long as mathematical and pre-mathematical target systems are ignored, researchers, textbook writers and teachers will hold on to the wrong assumption that children know nothing about mathematics when they enter school. What language teacher would assume that six-year-olds know nothing about the English or Spanish language when they enter schools? Perhaps teachers at special education institutes for the deaf-mute. As long as teachers keep thinking that children are mathematically deaf-mute, the teaching of mathematics, "Basic" or "New," is doomed to reach only the privileged upper ten per cent that manage to construct their own conceptual systems in spite of their teachers, disentangling by themselves the maze of symbolic systems disembodied of all their cultural and imaginative content.

Once they construct their conceptual systems, they can easily tune their receptors to the teacher's wavelength, keep learning mathematics, and start leaving their peers way behind. The teacher will teach the privileged students only, creating in the rest of them the "mathofobia" so well described by Papert (14).

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(14) PAPERT, S. Op. cit., Chapter 2: "Mathofobia: The Fear of Learning," pp. 38-54.

A serious analysis of every mathematical system encountered in the school curriculum should be carried out in detail before designing a new research program, a new textbook, and even a new lesson or school activity in mathematics. Here are some of the questions that are being researched here in Colombia, and that should be researched elsewhere, in order to make real this utopian goal

1. What experience with concrete systems, either mathematical or proto-mathematical, do children already have in their extraschool culture?

2. How can we delimit those culturally familiar target systems to establish their components, their procedures, their relations?

3. What words, images, symbols, signals, gestures, tokens... are used by children and adults to talk about the components, the procedures and the relations of such concrete target systems?

4. What transformations, actions, procedures or operators on those target systems constitute the raw materials for the construction of a new conceptual system?

5. What does it mean to compose, or successively apply those

operators to target components? What other conceptual combinations or operations can be performed on the operators that now form the components of the new conceptual system?

6. What relations hold among those operators?

7. What symbolic systems have been developed in the local culture, in the dominant middle-class culture, or in the mathematical literature to represent those operators, the procedures on them and their relations in the new conceptual system?

8. What procedures are used to manipulate and combine those symbols in the corresponding symbolic systems (algorithms)?

9. What relations hold among the symbols of each symbolic system, and among the symbols of different symbolic systems developed for the same conceptual system?

10. Do standard textbook notations and algorithms collide with the concrete, conceptual or symbolic handling of the culturally familiar target, conceptual or symbolic systems?

Thus, only two simple conceptual tools are used to

analyze the mathematical totalities. We could think of them as two triadic prisms that decompose the white light of the mathematical totality that is being analyzed; the first prism shows three levels of systems: concrete target systems, conceptual systems and symbolic systems; the second prism shows three different types of entities inside each system: components, procedures and relations. Once the whole picture is spread out in detail and understood in its complexity, educational research and curriculum design in that particular aspect of mathematics will have a definite framework, where the direct cultural dependence of symbolic systems, the indirect cultural dependence of conceptual systems, and, above all, the determining cultural dependence of concrete target systems will stand out.

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